PERGAMON

Applied Mathematics Letters 12 (1999) 69–76

Applied Mathematics Letters

Generalized Implicit Quasivariational Inequalities with Relaxed Lipschitz and Relaxed Monotone Mappings

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(Received March 1998; accepted July 1998)

Abstract—In this paper, we introduce and study a new class of generalized nonlinear implicit quasivariational inequalities for set-valued mappings and construct some new iterative algorithms. We prove the existence of solutions for this generalized nonlinear implicit quasivariational inequalities involving relaxed Lipschitz and relaxed monotone mappings and the convergence of iterative sequences generated by the algorithms.

Keywords—Implicit quasivariational inequality, Relaxed Lipschitz, Relaxed monotone.

1. INTRODUCTION

It is well known that variational inequality theory and complementarity problem theory are very powerful tools of the current mathematical technology. In recent years, classical variational inequality and complementarity problems have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, and transportation equilibrium and engineering sciences, etc. For details we refer to [1–19] and the references therein.

In a recent paper [1], Verma studied the solvability, based on an iterative algorithm, of a class of generalized nonlinear variational inequalities. However, from the assumptions of Theorem 3.1 of [1], we know that all the set-valued mappings in this theorem are single-valued mappings indeed. Moreover, from the proof of Theorem 3.1, we know $t < 1$.

Inspired and motivated by recent research works [1,4], in this paper, we introduce and study a class of generalized nonlinear implicit quasivariational inequalities for set-valued mappings and construct some new iterative algorithms. We prove the existence of solutions for this class of generalized nonlinear implicit quasivariational inequalities and the convergence of iterative sequences generated by the algorithms. Our results clarify, extend, and improve earlier and recent results of [1–3].

The authors are grateful to N.-J. Huang for his useful suggestions and stimulating discussions.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$, and inner product $\langle\cdot,\cdot\rangle$. Let K be a nonempty closed convex subset of H and P_K be the projection of H onto K.

Given single-valued mappings $f, m, F, G : H \to H$, and set-valued mappings $S, T, M, K : H \to 2^H$, we consider the following problem.

Find $x \in H$, $y \in Mx$, $w \in Sx$, $z \in Tx$ such that

$$f(x) \in K(y), \qquad \langle f(x) - (Fw - Gz), v - f(x)\rangle \geq 0, \qquad \forall v \in K(y). \tag{2.1}$$

Problem (2.1) is called the completely generalized strongly nonlinear implicit quasivariational inequality.

If M is an identity mapping and $K(x) = m(x) + K$, then problem (2.1) is equivalent to finding $x \in H$, $w \in Sx$, $z \in Tx$, such that

$$f(x) \in K(x), \qquad \langle f(x) - (Fw - Gz), v - f(x)\rangle \geq 0, \qquad \forall v \in K(x), \tag{2.2}$$

which is called the generalized strongly nonlinear implicit quasivariational inequality.

If F, G, and M are the identity mappings, and $K(x) = K$ for all $x \in H$, then problem (2.1) is equivalent to finding $x \in H$, $w \in Sx$, $z \in Tx$, such that

$$f(x) \in K, \qquad \langle f(x) - (w - z), v - f(x)\rangle \geq 0, \qquad \forall v \in K, \tag{2.3}$$

which is called the generalized variational inequality, considered by Verma [1].

It is clear that the completely generalized strongly nonlinear implicit quasivariational inequality problem (2.1) includes many kinds of quasivariational inequalities, variational inequality, complementarity, and quasicomplementarity problems as special cases, such as of [1–3].

DEFINITION 2.1. *A mapping $f : H \to H$ is said to be*

(i) *strongly monotone if there exists some $r > 0$ such that*

$$\langle fu_1 - fu_2, u_1 - u_2\rangle \geq r\|u_1 - u_2\|^2, \qquad \forall u_i \in H, \quad i = 1, 2;$$

(ii) *Lipschitz continuous if there exists some $s > 0$ such that*

$$\|fu_1 - fu_2\| \leq s\|u_1 - u_2\|, \qquad \forall u_i \in H, \quad i = 1, 2.$$

DEFINITION 2.2. *A set-valued mapping $S : H \to 2^H$ is said to be*

(i) *H-Lipschitz continuous if there exists some $\delta > 0$ such that*

$$H(Su_1, Su_2) \leq \delta\|u_1 - u_2\|, \qquad \forall u_i \in H, \quad i = 1, 2,$$

where $H(\cdot,\cdot)$ is the Hausdorff metric;

(ii) *relaxed Lipschitz with respect to a mapping $F : H \to H$ if there exists some $k \geq 0$ such that*

$$\langle Fw_1 - Fw_2, u_1 - u_2\rangle \leq -k\|u_1 - u_2\|^2, \qquad \forall u_i \in H,\ w_i \in Su_i, \quad i = 1, 2;$$

(iii) *relaxed monotone with respect to a mapping $G : H \to H$ if there exists some $c > 0$ such that*

$$\langle Gw_1 - Gw_2, u_1 - u_2\rangle \geq -c\|u_1 - u_2\|^2, \qquad \forall u_i \in H,\ w_i \in Su_i, \quad i = 1, 2.$$

3. ITERATIVE ALGORITHM

LEMMA 3.1. *(See [9].) If $K \subset H$ is a closed convex subset and $z \in H$ is a given point, then $u \in K$ satisfies the inequality*

$$\langle x - z, v - x \rangle \geq 0, \qquad \forall v \in K$$

iff

$$x = P_K z. \tag{3.1}$$

LEMMA 3.2. *(See [9].) The mapping P_K defined by (3.1) is nonexpansive, that is,*

$$\|P_K u - P_K v\| \leq \|u - v\|, \qquad \forall u, v \in H.$$

From Lemma 3.1, we have the following lemma.

LEMMA 3.3. *Elements $x \in H$, $y \in Mx$, $w \in Sx$, and $z \in Tx$ are a solution set of problem (2.1) iff $x \in H$, $y \in Mx$, $w \in Sx$, and $z \in Tx$ satisfy the equation for $0 < t < 1$,*

$$f(x) = P_{K(y)}[(1-t)f(x) + t(Fw - Gz)].$$

Based on Lemma 3.3, we are now to propose the following algorithm for problem (2.1).

ALGORITHM 3.1. *Let $K : H \to 2^H$ be a set-valued mapping such that for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H. Let $F, G, f : H \to H$, and $S, T, M : H \to CB(H)$, where $CB(H)$ is the family of all nonempty bounded closed subsets of H. For given $x_0 \in H$, we take $y_0 \in Mx_0$, $w_0 \in Sx_0$, and $z_0 \in Tx_0$, and let*

$$f(x_1) = P_{K(y_0)}[(1-t)f(x_0) + t(Fw_0 - Gz_0)].$$

Since $y_0 \in CB(H)$, $w_0 \in Sx_0 \in CB(H)$, $z_0 \in Tx_0 \in CB(H)$, by Nadler [20] there exist $y_1 \in Mx_1$, $w_1 \in Sx_1$, $z_1 \in Tx_1$ such that

$$\begin{aligned} \|y_0 - y_1\| &\leq (1+1)H(Mx_0, Mx_1), \\ \|w_0 - w_1\| &\leq (1+1)H(Sx_0, Sx_1), \\ \|z_0 - z_1\| &\leq (1+1)H(Tx_0, Tx_1). \end{aligned}$$

Let

$$f(x_2) = P_{K(y_1)}[(1-t)f(x_1) + t(Fw_1 - Gz_1)].$$

By induction, we can obtain sequences $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{f(x_n)\}$ as

$$\begin{gathered} f(x_{n+1}) = P_{K(y_n)}\left[(1-t)f(x_n) + t(Fw_n - Gz_n)\right], \\ y_n \in Mx_n, \quad \|y_n - y_{n+1}\| \leq \left(1 + (n+1)^{-1}\right) H(Mx_n, Mx_{n+1}), \\ w_n \in Sx_n, \quad \|w_n - w_{n+1}\| \leq \left(1 + (n+1)^{-1}\right) H(Sx_n, Sx_{n+1}), \\ z_n \in Tx_n, \quad \|z_n - z_{n+1}\| \leq \left(1 + (n+1)^{-1}\right) H(Tx_n, Tx_{n+1}), \\ n = 0, 1, 2, \dots, \end{gathered} \tag{3.2}$$

where $0 < t < 1$ is constant.

If $M : H \to H$ is the identity mapping and $K(x) = m(x) + K$, then from Algorithm 3.1 we have the following algorithm.

ALGORITHM 3.2. Suppose that $K \subset H$ is a closed convex subset, $F, G, f, m : H \to H$, and $S, T : H \to CB(H)$. For given $x_0 \in H$, we can obtain sequences $\{x_n\}$, $\{w_n\}$, $\{z_n\}$, $\{f(x_n)\}$ as

$$\begin{gathered} f(x_{n+1}) = m(x_n) + P_K[(1-t)f(x_n) + t(Fw_n - Gz_n) - m(x_n)], \\ w_n \in Sx_n, \quad \|w_n - w_{n+1}\| \le \left(1 + (n+1)^{-1}\right) H(Sx_n, Sx_{n+1}), \\ z_n \in Tx_n, \quad \|z_n - z_{n+1}\| \le \left(1 + (n+1)^{-1}\right) H(Tx_n, Tx_{n+1}), \\ n = 0, 1, 2, \dots, \end{gathered} \tag{3.3}$$

where $0 < t < 1$ is constant.

If $F, G, M : H \to H$ are identity mappings and $K(x) = K$, then from Algorithm 3.1 we have the following algorithm.

ALGORITHM 3.3. Suppose that $K \subset H$ is a closed convex subset, $f : H \to H$ and $S, T : H \to CB(H)$. For given $x_0 \in H$, we can obtain sequences $\{x_n\}$, $\{w_n\}$, $\{z_n\}$, and $\{f(x_n)\}$ as

$$\begin{gathered} f(x_{n+1}) = P_K[(1-t)f(x_n) + t(w_n - z_n)], \\ w_n \in Sx_n, \quad \|w_n - w_{n+1}\| \le \left(1 + (n+1)^{-1}\right) H(Sw_n, Sw_{n+1}), \\ z_n \in Tx_n, \quad \|z_n - z_{n+1}\| \le \left(1 + (n+1)^{-1}\right) H(Tz_n, Tz_{n+1}), \\ n = 0, 1, 2, \dots, \end{gathered} \tag{3.4}$$

where $0 < t < 1$ is constant.

REMARK 3.1.

(1) For appropriate and suitable choices of the mappings K, f, F, G, S, T, and M, a number of algorithms for variational inequality, quasivariational inequality, complementarity, and quasicomplementarity problems can be obtained as special cases of Algorithm 3.1.
(2) Algorithm 3.3 includes several known algorithms of [1–3] as special cases.

4. EXISTENCE AND CONVERGENCE

In this section, we prove the existence of solutions of problems (2.1)–(2.3) without compactness and the convergence of iterative sequence generated by Algorithms 3.1–3.3.

THEOREM 4.1. *Let F, G, $f : H \to H$ be Lipschitz continuous with Lipschitz constants ξ, η, and s, respectively, and f be strongly monotone with constant r. Let M, S, $T : H \to CB(H)$ be H-Lipschitz continuous with H-Lipschitz constants γ, h, d, respectively, S relaxed Lipschitz with respect to F with constant k and T be relaxed monotone with respect to G with constant c. Let $K : H \to 2^H$ be a set-valued mapping such that for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H. Suppose there exists a constant $\mu > 0$ such that for each $x, y, z \in H$,*

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \le \mu\|x - y\|. \tag{4.1}$$

If the following conditions hold,

$$\begin{gathered} \left| t - \frac{1 + k - c + p(r - p - \mu\gamma)}{1 + 2(k - c) + (\xi h + \eta d)^2 - p^2} \right| \\ < \frac{\sqrt{(1 + k - c + p(r - p - \mu\gamma))^2 - (1 + 2(k - c) + (\xi h + \eta d)^2 - p^2)(1 - (r - p - \mu\gamma)^2)}}{1 + 2(k - c) + (\xi h + \eta d)^2 - p^2}, \end{gathered} \tag{4.2}$$

$$r - p - \mu\gamma > tp, \qquad 1 + 2(k - c) + (\xi h + \eta d)^2 > p^2 \tag{4.3}$$

and one of the following conditions holds,

$$r - p - \mu\gamma > 1, \qquad 1 + k + p(r - p - \mu\gamma) \le c, \tag{4.4}$$

$$\xi h + \eta\, d < r - \mu\gamma, \qquad \xi h + \eta\, d \le \sqrt{p(r - \mu\gamma) + c - k}, \tag{4.5}$$

$$\begin{gathered} 0 < 1 + k - c + p(r - p - \mu\gamma) < 1 + 2(k - c) + (\xi h + \eta\, d)^2 - p^2 \\ (1 + k - c + p(r - p - \mu\gamma))^2 > \left(1 + 2(k - c) + (\xi h + \eta\, d)^2 - p^2\right)\left(1 - (r - p - \mu\gamma)^2\right), \end{gathered} \tag{4.6}$$

where $p = \sqrt{1 - 2r + s^2}$, then there exist $x \in H$, $y \in Mx$, $w \in Sx$, and $z \in Tx$ which are a solution of problem (2.1), and

$$x_n \to x, \qquad y_n \to y, \qquad w_n \to w, \qquad z_n \to z, \qquad f(x_n) \to f(x), \qquad n \to \infty,$$

where $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{f(x_n)\}$ are defined in Algorithm 3.1.

PROOF. From Algorithm 3.1, Lemma 3.2, and (4.1), we have

$$\begin{aligned} \|f(x_{n+1}) - f(x_n)\| &\le \|P_{K(y_n)}(A(x_n)) - P_{K(y_n)}(A(x_{n-1}))\| \\ &\quad + \|P_{K(y_n)}(A(x_{n-1})) - P_{K(y_{n-1})}(A(x_{n-1}))\| \\ &\le \|A(x_n) - A(x_{n-1})\| + \mu\|y_n - y_{n-1}\| \\ &\le \mu\|y_n - y_{n-1}\| + \|(1 - t)(x_n - x_{n-1} - (f(x_n) - f(x_{n-1})))\| \\ &\quad + \|(1 - t)(x_n - x_{n-1}) + t(Fw_n - Fw_{n-1}) - t(Gz_n - Gz_{n-1})\|, \end{aligned} \tag{4.7}$$

where $A(x_n) = (1 - t)f(x_n) + t(Fw_n - Gz_n)$. By the Lipschitz continuity and strong monotonicity of f, we obtain

$$\|x_{n+1} - x_n\| \le \left(\frac{1}{r}\right)\|f(x_{n+1}) - f(x_n)\|, \tag{4.8}$$

$$\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \le \left(1 - 2r + s^2\right)\|x_n - x_{n-1}\|^2. \tag{4.9}$$

Since M, S, T are H-Lipschitz continuous and F, G are Lipschitz continuous, we get

$$\|y_n - y_{n-1}\| \le \gamma\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|, \tag{4.10}$$

$$\|Fw_n - Fw_{n-1}\| \le \xi\|w_n - w_{n-1}\| \le \xi h\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|, \tag{4.11}$$

$$\|Gz_n - Gz_{n-1}\| \le \eta\|z_n - z_{n-1}\| \le \eta\, d\left(1 + \frac{1}{n}\right)\|x_n - x_{n-1}\|. \tag{4.12}$$

Further, since S is relaxed Lipschitz and T is relaxed monotone, we have

$$\begin{aligned} &\|(1 - t)(x_n - x_{n-1}) + t(Fw_n - Fw_{n-1}) - t(Gz_n - Gz_{n-1})\|^2 \\ &\quad = (1 - t)^2\|x_n - x_{n-1}\|^2 + 2t(1 - t)\langle Fw_n - Fw_{n-1}, x_n - x_{n-1}\rangle \\ &\qquad - 2t(1 - t)\langle Gz_n - Gz_{n-1}, x_n - x_{n-1}\rangle + t^2\|Fw_n - Fw_{n-1} - (Gz_n - Gz_{n-1})\|^2 \\ &\quad \le \left[(1 - t)^2 - 2t(1 - t)(k - c) + t^2\left(1 + \frac{1}{n}\right)^2(\xi h + \eta\, d)^2\right]\|x_n - x_{n-1}\|^2. \end{aligned} \tag{4.13}$$

From (4.7)–(4.13), it follows that

$$\|x_{n+1} - x_n\| \le \theta_n\|x_n - x_{n-1}\|, \tag{4.14}$$

where

$$\theta_n = \left(\frac{1}{r}\right)\left(\mu\gamma\left(1+\frac{1}{n}\right) + (1-t)p + \sqrt{(1-t)^2 - 2t(1-t)(k-c) + t^2\left(1+\frac{1}{n}\right)^2 (\xi h + \eta d)^2}\right),$$

and $p = \sqrt{1-2r+s^2}$. Letting

$$\theta := \left(\frac{1}{r}\right)\left(\mu\gamma + (1-t)p + \sqrt{(1-t)^2 - 2t(1-t)(k-c) + t^2(\xi h + \eta\, d)^2}\right),$$

we know that $\theta_n \searrow \theta$. It follows from (4.2),(4.3) and one of (4.4)–(4.6) that $\theta < 1$. Hence $\theta_n < 1$, for n sufficiently. Therefore, (4.14) implies that $\{x_n\}$ is a Cauchy sequence in H and we can suppose that $x_n \to x \in H$. Since f is Lipschitz continuous,we have $f(x_n) \to f(x)$ as $n \to \infty$.

Now we prove that $y_n \to y \in Mx$, $w_n \to w \in Sx$, $z_n \to z \in Tx$. In fact, it follows from Algorithm 3.1 that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le \left(1+\frac{1}{n}\right)\gamma\|x_n - x_{n-1}\|,\\
\|w_n - w_{n-1}\| &\le \left(1+\frac{1}{n}\right)h\|x_n - x_{n-1}\|,\\
\|z_n - z_{n-1}\| &\le \left(1+\frac{1}{n}\right)d\|x_n - x_{n-1}\|,
\end{aligned}$$

that is, $\{y_n\}$, $\{w_n\}$, and $\{z_n\}$ are also Cauchy sequences in H. Let $y_n \to y$, $w_n \to w$, $z_n \to z$ as $n \to \infty$. Further, we have

$$\begin{aligned}
d(w, Sx) = \inf\{\|w-u\| : u \in Sx\} &\le \|w - w_n\| + d(w_n, Sx)\\
&\le \|w - w_n\| + H(Sx_n, Sx)\\
&\le \|w - w_n\| + h\|x_n - x\| \to 0, \qquad n \to \infty.
\end{aligned}$$

Hence, $w \in Sx$. Similarly, $y \in Mx$, $z \in Tx$. This completes the proof.

From Theorem 4.1, we can get the following results.

THEOREM 4.2. *Let F, G, f, $m : H \to H$ be Lipschitz continuous with Lipschitz constants ξ, η, s, and μ, respectively, and f be strongly monotone with constant r. Let S, $T : H \to CB(H)$ be H-Lipschitz continuous with H-Lipschitz constants h, d, respectively, S relaxed Lipschitz with respect to F with constant k and T be relaxed monotone with respect to G with constant c. If the following conditions hold,*

$$\left|t - \frac{1+k-c+p(r-p-2\mu)}{1+2(k-c)+(\xi h+\eta\, d)^2 - p^2}\right|$$
$$< \frac{\sqrt{(1+k-c+p(r-p-2\mu))^2 - (1+2(k-c)+(\xi h+\eta\, d)^2 - p^2)\,(1-(r-p-2\mu)^2)}}{1+2(k-c)+(\xi h+\eta\, d)^2 - p^2}, \tag{4.15}$$

$$r - p - 2\mu > tp, \qquad 1+2(k-c)+(\xi h+\eta\, d)^2 > p^2 \tag{4.16}$$

and one of the following conditions holds,

$$r-p-2\mu > 1, \qquad 1+k+p(r-p-2\mu) \le c, \tag{4.17}$$

$$\xi h + \eta\, d < r - 2\mu, \qquad \xi h + \eta\, d \le \sqrt{p(r-2\mu)+c-k}, \tag{4.18}$$

$$
\begin{aligned}
&0 < 1 + k - c + p(r - p - 2\mu) < 1 + 2(k - c) + (\xi h + \eta d)^2 - p^2 \\
&(1 + k - c + p(r - p - 2\mu))^2 > \left(1 + 2(k - c) + (\xi h + \eta d)^2 - p^2\right)\left(1 - (r - p - 2\mu)^2\right),
\end{aligned} \tag{4.19}
$$

where $p = \sqrt{1 - 2r + s^2}$, then there exist $x \in H$, $w \in Sx$, and $z \in Tx$ which are a solution of problem (2.2), and

$$
x_n \to x, \qquad w_n \to w, \qquad z_n \to z, \qquad f(x_n) \to f(x), \qquad n \to \infty,
$$

where $\{x_n\}$, $\{w_n\}$, $\{z_n\}$, $\{f(x_n)\}$ are defined in Algorithm 3.2.

THEOREM 4.3. *Let $f : H \to H$ be strongly monotone and Lipschitz continuous with corresponding constants $r > 0$ and $s > 0$. Let $S : H \to CB(H)$ be relaxed Lipschitz with constant $k \geq 0$ and H-Lipschitz continuous with constant $h > 0$. Let $T : H \to CB(H)$ be relaxed monotone with constant $c > 0$ and H-Lipschitz continuous with constant $d > 0$. If the conditions hold,*

$$
\begin{aligned}
&\left| t - \frac{1 + k - c + p(r - p)}{1 + 2(k - c) + (h + d)^2 - p^2} \right| \\
&\quad < \frac{\sqrt{(1 + k - c + p(r - p))^2 - \left(1 + 2(k - c) + (h + d)^2 - p^2\right)\left(1 - (r - p)^2\right)}}{1 + 2(k - c) + (h + d)^2 - p^2},
\end{aligned} \tag{4.20}
$$

$$
r - p > tp, \qquad 1 + 2(k - c) + (h + d)^2 > p^2 \tag{4.21}
$$

and one of the following conditions holds,

$$
r - p > 1, \qquad 1 + k + p(r - p) \leq c, \tag{4.22}
$$

$$
h + d < r, \qquad h + d \leq \sqrt{pr + c - k}, \tag{4.23}
$$

$$
\begin{aligned}
&0 < 1 + k - c + p(r - p) < 1 + 2(k - c) + (h + d)^2 - p^2 \\
&(1 + k - c + p(r - p))^2 > \left(1 + 2(k - c) + (h + d)^2 - p^2\right)\left(1 - (r - p)^2\right),
\end{aligned} \tag{4.24}
$$

where $p = \sqrt{1 - 2r + s^2}$, then there exist $x \in H$, $w \in Sx$, and $z \in Tx$ which are a solution of problem (2.3), and

$$
x_n \to x, \qquad w_n \to w, \qquad z_n \to z, \qquad f(x_n) \to f(x), \qquad n \to \infty,
$$

where $\{x_n\}$, $\{w_n\}$, $\{z_n\}$, $\{f(x_n)\}$ are defined in Algorithm 3.3.

REMARK 4.1. For a suitable choice of the mappings K, f, F, G, S, T, and M, we can obtain several known results in [1–3] as special cases of Theorem 4.1.

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